MATH :- GE-04

Q) prove that if f is bounded and integrable in the interval [a, b] then there exist a number μ lies in interval such that

$$\int_a^b f(x)\,dx = \mu(a-b)$$

Sol:- we know that
if a function f is bounded in [a, b] then f is said to riemann integrable if

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

where

$$\overline{\int_a^b} f = \text{upper riemann integral.}$$

$$\underline{\int_a^b} f = \text{lower " "}$$

Hence

$$\inf[U(f, P_0), U(f, P_1) \ldots U(f, P_n)] = \sup[L(f, P_0), L(f, P_1) \ldots L(f, P_n)]$$

for some value p and q

$$\inf[U(f, P_0), U(f, P_1) \ldots U(f, P_n)] = \frac{b-a}{p}\sum_{i=1}^{n} f(x_i)$$

$$\left[\text{as } \Delta x = \frac{b-a}{p}\right]$$

~~sup[L(f, P₀), L(f, P₁) ...... L(f, Pₙ)~~)

$$= \frac{b-a}{p} \cdot M \qquad \left(M = \sum_{i=1}^{n} f(x_i) \right)$$

$$\sup [L(f, P_0), L(f, P_1) \ldots\ldots L(f, P_n)] = \frac{b-a}{p} \sum_{i=1}^{n} f(x_i)$$

$$= \frac{b-a}{q} \cdot m \qquad \left(m = \sum_{i=1}^{n} f(x_i) \right)$$

as $\Delta x = \frac{b-a}{q}$

for condition of integrability upper riemann integral = Lower Riemann integral

$\inf (U(f, P_i)) = \sup (L(f, P_i))$ for all i

$$\frac{\cancel{b-a}}{p} \cdot M = \frac{\cancel{b-a}}{q} \cdot m$$

$$\frac{M}{p} = \frac{m}{q} = \mu \; [\text{let}]$$

Hence $$\overline{\int_a^b} f = \underline{\int_a^b} f = \int_a^b f = \mu(b-a)$$

② Necessary and sufficient condition for riemann integral:

Sol:- Theorem :- If f be a bounded function on $[a, b]$ then f is R-intigrable if for $\varepsilon > 0$ $\exists$ a ~~partion~~ partition P over $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

$10+15\times 9+6\div 3$ $\cancel{10+135+2}$ $25-9$ $10+15\times 9-6\div 3$ $10+15-18$ $10-15+9\times 6\div 3$ $10-15+9\times 6\div 2$

proof :- Necessary part

let f is R- integrable on $[a,b]$, Then

$$\underline{\int_a^b} f(x) = \overline{\int_a^b} f(x) = \int_a^b f(x)\,dx$$

Now from darbox theorem, we know that $\forall\ \varepsilon > 0$

$$L(P,f) > \underline{\int_a^b} f(x)\,dx - \varepsilon/2 \quad \text{—} \ (1) \qquad [\varepsilon \to \varepsilon/2]$$

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon/2 \quad \text{—} \ (ii)$$

on multiplying eq (1) with -1

$$-L(P,f) < -\underline{\int_a^b} f(x)\,dx + \varepsilon/2 \quad \text{—} \ (iii)$$

adding eq (ii) and eq (iii)

$$U(P,f) - L(P,f) < \varepsilon \quad \text{proved}$$

proof :- Sufficient part

let $U(P,f) - L(P,f) < \varepsilon$

$$\underline{\int_a^b} f(x)\,dx = \text{l.u.b } L(P,f)$$

$$\overline{\int_a^b} f(x)\,dx = \text{g.l.b } U(P,f)$$

We know that

$$\underline{\int_a^b} f(x)\,dx \geq L(P,f) \quad \text{— (iv)}$$

$$\overline{\int_a^b} f(x)\,dx \leq U(P,f) \quad \text{— (v)}$$

on multiplying eqn (iv) with -1

$$-\underline{\int_a^b} f(x)\,dx < -L(P,f) \quad \text{— (vi)}$$

Adding (v) & (vi)

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(P,f) - L(P,f)$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq \varepsilon$$

$$\overline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} f(x)\,dx \quad \text{----- (vii)}$$

Now we know that

lower R-integrable $\leq$ upper R-integrable

$$\overline{\int_a^b} f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx \quad \text{------ (viii)}$$

from (vii) & (viii)

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Hence f is R-integrable on $[a,b]$

(3) prove that if f is integrable in $[a,b]$ then f^2 is also integrable.

Sol:- $f \in R[a,b]$

$\Rightarrow |f| \in R[a,b]$

f is bounded on $[a,b]$
$|f|$ is also bounded on $[a,b]$
$|f|^2 = f^2$ is also bounded on $[a,b]$

$f \in [a,b] \Rightarrow f$ is bounden the $\exists$ a partition $P \in \phi[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

let infimum of f is $= m_r$
supremum " $= M_r$

we know that

$$U(P,f) - L(P,f) < \frac{\varepsilon}{2m} \qquad \left[\because \varepsilon = \frac{\varepsilon}{2m}\right]$$

$$\Rightarrow \sum_{r=1}^{n} M_r - \sum_{r=1}^{n} m_r < \frac{\varepsilon}{2m}$$

we have request to prove $U(P, f^2) - L(P, f^2) < \varepsilon$

$\because$ f^2 is bounded on $[a, b]$

infimum of f^2 = m_r^2
sup " = M_r^2

Now L.H.S = $U(P, f^2) - L(P, f^2)$

$$\leq \sum_{r=1}^{n} M_r^2 - \sum_{r=1}^{n} m_r^2$$

$$\leq \sum_{r=1}^{n} (M_r^2 - m_r^2)$$

$$\leq \sum_{r=1}^{n} (M_r + m_r)(M_r - m_r)$$

$$\leq \frac{\varepsilon}{2m} \cdot 2m$$

$\therefore$ $U(P, f^2) - L(P, f^2) < \varepsilon$

$\therefore$ f^2 is integrable on $[a, b]$

(4) State and prove cauchy Riemann equation

Sol:- ~~Statement~~:- If a function $f(z) = u(x, y) + iv(x, y)$ is diffrentiable at any point $z = x + iy$. Then the partial derivation u_x, u_y, v_x, v_y should exist and satisfy the equation

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

proof :- $u_x = v_y$ and $u_y = v_x$

~~Since oflez is diffrentiable at~~

Since $f(z) = u(x,y) + i v(x,y)$ is diffrentiable at point z

Then

$$\lim_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\} \quad \text{— (I)}$$

must tends to a unique limit $(\Delta z \to 0)$

$\because$ $z = x + iy$

$\Delta z = \Delta x + i\Delta y$

$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$

$\therefore f(z) = u(x,y) + iv(x,y)$

Eq (I) become

$$\frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

$$= \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + iv(x+\Delta x, y+\Delta y) - iv(x,y)}{\Delta x + i\Delta y} \quad \text{— (II)}$$

case: 1 Δz to be wholy real so that $\Delta y = 0$

Then eq^n (II) becomes

$$\lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{iv(x+\Delta x, y) - iv(x,y)}{\Delta x} \right] \quad \text{— (III)}$$

Since $f(z)$ is diffrentiable, the limit given eq (iii) should also exist. i.e partial derivative $\frac{\partial u}{\partial x} = (u_x)$

and $\frac{\partial v}{\partial x} = v_x$ should 1.

$$Lt = u_x + iv_x \quad \text{—} \quad (4)$$

case:- 2 we take Δz to be wholly imaginary i.e $\Delta z \to 0$ so that $\Delta x = 0$ then eq (ii) becomes

$$\lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{iv(x, y+\Delta y) - iv(x,y)}{i\Delta y} \right]$$

$$\lim_{\Delta y \to 0} \left[-i\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -iu_y + v_y \quad \text{----} \quad (v)$$

The limit (4) and (5) must be identical

$$\therefore u_x + iv_x = v_y - iu_y$$

$$u_x = v_y, \quad v_x = -u_y$$

$$u_x = v_y, \quad u_y = -v_y \quad \text{—} \quad (vi)$$

Hence equation (vi) is known as cauchy-Riemann equations

5. Explain the milne-thomson method to construct an analytic function $f(z) = u + iv$ when the real part u is given

Sol: let $u = x^3 - 3xy^2$ It is real part of analytic function

we have to find $f(z)$

$\therefore u = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ | $\frac{\partial u}{\partial y} = -6y$

$\phi_1(z,0) = 3z^2 - 0$ | $\phi_2(z,0) = 0$

$= 3z^2$

By using milne-Thomson method

$f(z) = \int \phi_1(z,0)\,dz - i \int \phi_2(z,0)\,dz + c$

$= \int 3z^2\,dz - i \int 0\,dz + c$

$f(z) = z^3 + c$ Ans

6. find an analytic function $F(z) = u + iv$ and express it in term of z if $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Sol: let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 6x$

$\phi_1(z,0) = 3z_1^2 + 6z_1$ — (I)

$\frac{\partial v}{\partial y} = -6xy - 6y$

$\phi_2(z,0) = 0$ — (II)

By using milne-thomson method

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$

$= \int (3z^2+6z)\,dz - i\cdot 0 + c$

$= z^3 + 3z^2 + c$ Ans

(7) Show that the function $F(z) = e^x \cos y$ is harmonic and find its conjugate.

Sol:- Given:- $F(z) = e^x \cos y$

$\frac{\partial v}{\partial x} = e^{1x}\cos(y)$

$\frac{\partial^2 v}{\partial x^2} = e^{1x}\cos(y)$

$\frac{\partial v}{\partial y} = -e^{1x}\sin(y)$

$\frac{\partial^2 v}{\partial y^2} = -e^{1x}\cos(y)$

$3z^2dz + 6zdz$

$\cancel{3}\frac{z^3}{\cancel{3}} + \cancel{6}\frac{z^2}{\cancel{2}}$

$\frac{z^3}{?}$

$\partial^2 u\, dx^2 + \partial^2 u\, dy^2 = 0$

$\therefore$ u is the harmonic function.

We know that

$U_x = V_y$ (from cauchy-Riemann eq)

$\therefore V_y = \cancel{e^{x}\cos(y)}\ e^{\cos y}$

$\cancel{f(z)} = \cancel{u+iv} = \cancel{e^x\cos(y)} + \cancel{i(e^x\cos(y))} = \cancel{e^z + c}$

$\cancel{f(z)} = \cancel{e(x+iy)}$

$f(z) = u+iv = \left(e^{x\cos(y)} + i e^{x\cos(y)}\right) = e^z + c$

$f(z) = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\cos(y))$

$= e^{x\cos y} + i e^{x\cos(y)}$

8) prove that $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ does not exist

Sol:- let $z = x+iy$

$\bar{z} = x - iy$

$f(z) = \dfrac{x-iy}{x+iy} = \dfrac{\bar{z}}{z}$

Suppose that $\lim\limits_{z\to 0} f(z)$ exist, then approaching (0,0) through any direction the value of $\lim\limits_{z\to 0} f(z)$ must be same.

let us consider to the origin (0,0) therom the x-axis (x,0) then

$$f(z) = \frac{x - i\cdot 0}{x + i\cdot 0} = 1$$

$$f(z) = 1$$

so $\lim_{z\to 0} f(z) = 1$

Now consider the origin (0,0) through the y-axis (0,y) then

$$f(z) = \frac{0 - iy}{0 + iy} = -1$$

$$f(z) = -1$$

$\because$ f(z) have not unique value at $z \to 0$

Hence $\lim_{z\to 0} f(z) = \frac{\bar{z}}{z}$ does not exist.

Q) prove that if $f(z) = z^2$ then $f'(z) = 2z$

Sol:- $f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$\because$ This limit exist, Then the function f is said to be diffrentiable at z_0 when it derivative at z_0 exist

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \qquad \left\{ \Delta w = f(z + \Delta z) - f(z) \right.$$

$f(z) = z^2$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{z^2 + 2z\cdot\Delta z + \Delta z^2 - z^2}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(2z + \Delta z)}{\cancel{\Delta z}}$$

$$= 2z + 0$$

$$= 2z$$

$f'(z) = 2z$ proved.

10) prove that if $f(z) = e^z$, Then $f'(z) = e^z$

Sol:- Given $f(z) = e^z$

$$f(z) = e^{x+iy}$$

$$u(x,y) + iv(x,y) = e^x \cdot e^{iy}$$

$$u(x,y) + iv(x,y) = e^x [\cos y + i\sin y]$$

$$u(x,y) + iv(x,y) = e^x \cos y + ie^x \sin y$$

$$u(x,y) = e^x \cos y$$

$$v(x,y) = e^x \sin y$$

The function is analytic, If it statisfy Cauchy Riemann equation

$\because$ C-R eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = e^x \cos y \quad \text{— (I)}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \sin y \quad \text{— (II)}$$

$\because$ $f(z) = e^z$

$$f(z) = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$= e^x(\cos y + i\sin y)$$

$$f(z) = e^x \cos y + ie^x \sin y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + ie^x \sin y$$

$$= e^x(\cos y + i\sin y)$$

$$= e^x \cdot e^{iy}$$

$$= e^{(x+iy)}$$

$$= e^z$$ proved

12) let $A = \{1, 2, 3, 4, 5, 6\}$ and R is relation "x divide y". Find R^{-1}

Sol: let $A = \{1, 2, 3, 4, 5, 6\}$

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$
$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$
$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$
$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)$

$\because R = \{(x,y) \in A \times A : y/x\}$

$\therefore R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,2)\ (2,4)\ (2,6)$
$(3,3)\ (3,6)$
$(4,4)$
$(5,5)$
$(6,6)\}$

$R^{-1} = \{(1,1)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1)$
$(2,2)\ (4,2)\ (6,2)\ (3,3)\ (6,3)$
$(4,4)\ (5,5)\ (6,6)\}$

17) prove that two equivalance class are equal or disjoint.

Sol:- R- equivalance relation on X
$[x]$ and $[y]$

let $[x] \cap [y] \neq \phi$

$a \in [x] \cap [y]$

$\Rightarrow a \in [x]$ and $a \in [y]$

$\Rightarrow aRx$ and $aR'y$

$\Rightarrow xRa$ and aRy

$\Rightarrow xRy$

$\Rightarrow x \in [y]$

$x = y$ hence equivalance class are equal.

18) Give an example of relation which is reflexive transitive but not symmetric.

Sol:

Reflexive	Symmetric	Transitive
$A = \{a, b\}$ $R = \{(a,a)\ (b,b)\}$	$(a,b) \in R$ $(b,a) \in R$	$(a,b) \in R,\ (b,c) \in R$ $\Rightarrow (a,c) \notin R$

$A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (2,2), (3,3), (4,4)\ (5,5)\ (1,2), (2,3), (1,3)\}$

Relation R is ~~symmetric~~ Reflexive and transitve but

$(1,2) \in R,\ (2,1) \notin R$

hence R is not symmetric.

Q) Explain the example that a system of linear equation has infinite solution.

Sol:- When the two lines are the same line then the system should have infinite solution. It means that if the system of equation has an infinite number of solution, then the system is said to be consistent.

Thus, suppose we have two equations in two variable as follows:-

$$a_1x + b_1y = c_1 \quad \text{—— (i)}$$
$$a_2x + b_2y = c_2 \quad \text{—— (ii)}$$

The given equation are consistent and denpendent and have infinitely many solution if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Show that the following system of equation has infinite solution:-

$$2x + 5y = 10 \quad \text{and} \quad 10x + 25y = 50$$

By comparing with linear system we get

$a_1 = 2$, $b_1 = 5$, $c_1 = 10$
$a_2 = 10$, $b_2 = 25$, $c_2 = 50$

Now the ratio are

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{\cancel{2}^{1}}{\cancel{10}_{5}} = \frac{\cancel{5}^{1}}{\cancel{25}_{5}} = \frac{10}{50}$$

$\because \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

The given equation has infinitely many solution.

Q) Show that $f(z) = |z|^2$ is nowhere analytic.

Sol:- Let $z = x + iy$

we have that

$$f(z) = |z|^2 = z \cdot \bar{z} = x^2 + y^2 \quad - (1)$$

This show that is real valued function

we can rewrite above eq - (1)

$$f(z) = x^2 + y^2 + i \cdot 0$$

$$u(x, y) = x^2 + y^2$$

$$v(x, y) = 0$$

$$f(x, y) = u(x, y) + i \cdot v(x, y)$$

The function f is continuous because u, v are continuous.

Cauchy-Riemann holds ~~the~~ at the origin

$$u_x = 2x, \quad u_y = 2y$$
$$u_x = v_y, \quad u_y = -v_x$$

$x = 0, y = 0 \Rightarrow z = 0$, f is only diffrentiable at origin.

$\therefore f(z) = |z|^2$ is nowhere analytic.